

## Topic Notes: Decrease and Conquer

Our next class of algorithms are the *decrease and conquer* group.

The idea here is that we solve a problem by:

1. Reducing the problem to a smaller instance
2. Solving the smaller instance
3. Modifying the smaller instance solution to be a solution to the original

The main variations on this are:

- Decrease problem size by a constant (often by 1)
- Decrease problem size by a constant factor (often by half)
- Variable size decrease

For example, consider variations on how to compute the value  $a^n$ .

The brute-force approach would involve applying the definition and multiplying  $a$  by itself,  $n$  times.

A *decrease-by-one* approach would reduce the problem to computing the result for the problem of size  $n - 1$  (computing  $a^{n-1}$ ) then modifying that to be a solution to the original (by multiplying that result by  $a$ ).

A *decrease-by-constant-factor* (2, in this case) approach would involve computing  $a^{\frac{n}{2}}$  and multiplying that result by itself to compute the answer. The only complication here is that we have to treat odd exponents specially, leading to a rule:

$$a^n = \begin{cases} (a^{\frac{n}{2}})^2 & \text{if } n \text{ is even} \\ (a^{\frac{n-1}{2}})^2 \cdot a & \text{if } n > 1 \text{ and odd} \\ a & \text{if } n = 1 \end{cases}$$

This approach will lead to  $O(\log n)$  multiplications.

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## Insertion Sort

Our decrease-by-one approach to sorting is the *insertion sort*.

The insertion sort sorts an array of  $n$  elements by first sorting the first  $n - 1$  elements, then inserting the last element into its correct position in the array.

```
ALGORITHM INSERTIONSORT( $A$ )
  //Input: an array  $A[0..n - 1]$ 
  for  $i \leftarrow 0..n - 1$  do
     $v \leftarrow A[i]$ 
     $j \leftarrow i - 1$ 
    while  $j \geq 0$  and  $A[j] < v$  do
       $A[j + 1] \leftarrow A[j]$ 
       $j \leftarrow j - 1$ 
     $A[j + 1] \leftarrow v$ 
```

We will examine this one in more detail on the handout.

Of the simple sorting algorithms (bubble, selection, insertion), insertion sort is considered the best option in general.

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## Recursive Insertion Sort

As we move into our decrease and conquer and moreso with divide and conquer, we will often find it convenient to consider algorithms in recursive formulations. Let's do that with insertion sort.

Think about what the iterative solution did: it built up larger and larger sorted subarrays by inserting another value into its correct position. After  $n - 1$  such steps, the entire array is sorted.

To think about this problem recursively, we will, in some sense, think about the problem backwards. Assuming we can obtain a sorted array of size  $k - 1$  (recursive step), we simply need to insert one element into its correct position to get a sorted array of size  $k$ .

```
ALGORITHM INSERTIONSORT( $A$ )
  //Input: an array  $A[0..n - 1]$ 
   $recInsertionSort(A, n - 1)$ 

ALGORITHM RECINSERTIONSORT( $A, max$ )
  //Input: an array  $A[0..n - 1]$ 
  //Input: upper index limit to sort  $max$ 
  // Base case: a 1-element array
  if  $max = 0$  then
    return
  // Recursive case: sort first  $max - 1$ 
   $RecInsertionSort(A, max - 1)$ 
  // now insert  $max$ 'th in correct location
   $v \leftarrow A[max]$ 
   $j \leftarrow max - 1$ 
  while  $j \geq 0$  and  $A[j] > v$  do
     $A[j + 1] \leftarrow A[j]$ 
```

$$\begin{aligned}j &\leftarrow j - 1 \\ A[j + 1] &\leftarrow v\end{aligned}$$

This algorithm performs the same steps as the iterative version, but how could we analyze this? The recursive step doesn't lend itself well to setting up summations. Instead, we will set up and solve a recurrence.

Before solving this one, we will look at a few recurrences in the topic notes on Analysis of Recurrences (which are also part of our current handout), then will complete our analysis of recursive insertion sort, also in the current handout.

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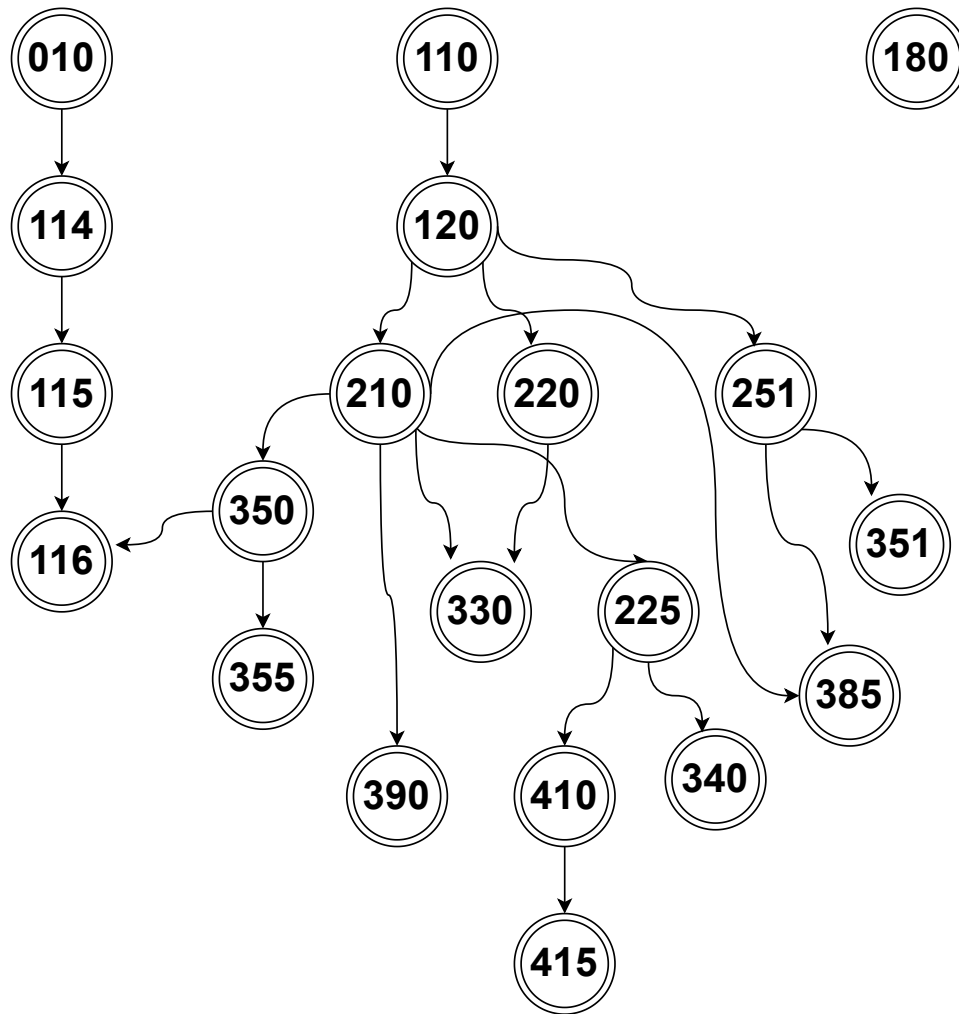
## Topological Sort

Our next algorithm operates on a special class of graph structures: *directed acyclic graphs*, or *dags*.

Dags naturally arise in many problems, but we will assume that we are given a dag and wish to perform a *topological sort* of the graph vertices. A topological sort of a dag is an ordering of the vertices such that for every edge in the dag, the starting vertex of the edge is listed before the ending vertex.

Note that it makes no sense to attempt a topological sort if the graph is either undirected or if it has a cycle.

One example of a dag might be a course prerequisite graph.



One algorithm to compute a topological sort of a dag is based on a depth-first traversal of the graph is described in the text.

We will look at a second option (also in the text) called the *source removal algorithm*.

The algorithm proceeds as follows:

**ALGORITHM** SOURCEREMOVAL( $G$ )

//Input: a dag  $G = \{V, E\}$

$o \leftarrow$  a new empty list

**while**  $\exists$  a source vertex in  $G$  **do**

$v \leftarrow$  a source vertex from  $G$

$o.append(v)$

$G.remove(v)$  // also removes all incident edges

**if**  $G$  is empty **then**

**return**  $o$

**else**

**flag error** //  $G$  was not a dag

## Generating Combinatorial Objects

One of the things we saw in the “brute force” algorithm discussion was that we sometimes need to enumerate all possible solutions for a given problem.

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### Permutations

You already implemented one mechanism for generating all permutations of a list of items for a problem set.

The text describes two additional methods for enumerating permutations that we will not discuss in class. It is worth reading about them.

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### Subsets

We saw the need to generate all possible subsets (*i.e.*, a *power set*) when discussing the knapsack problem.

The text describes four approaches.

For the first, we will use a direct decrease and conquer approach. We will consider each element in the set, and generate all of the sets that do not have that element. Then, the subsets are exactly those plus the same group of subsets but with this element added in. See the table in Figure 4.10, p. 147, for an example.

A quite clever method (and your instructor’s favorite mechanism) involves using an integer value, treating its bottom  $n$  bits as Boolean values indicating whether one of the elements is in the power set or not. The advantage of this is that the loop to visit all subsets becomes a `for` loop, counting from 0 to  $2^n - 1$ . And inside the loop, we just find the “1” bits and treat the corresponding elements as being in the set, or not, as appropriate. This generates a “lexicographic order”.

Another alternative is to generate the “squashed order”, where we first generate the empty set, then the singletons in order, then the two-element subsets, etc. This one is left as an exercise.

Finally, there is the Gray code ordering, where each subset in the sequence differs from the next by the addition or removal of a single element.

See the algorithm on p. 148.

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## Decrease-by-a-constant-factor

### Binary Search

We next briefly recall the idea of a *binary search* – the efficient procedure for searching for an item in a sorted array.

A binary search is often treated as an example of a divide and conquer algorithm – our next major group of algorithms, but Levitin treats it as an example of a *decrease-by-a-constant-factor algo-*

*arithm.*

Here is a nonrecursive algorithm that implements binary search:

```

ALGORITHM BINARYSEARCH( $A, K$ )
  //Input: a sorted array  $A[0..n - 1]$ 
  //Input: search key  $K$ 
   $l \leftarrow 0$ 
   $r \leftarrow n - 1$ 
  while  $l \leq r$  do
     $m \leftarrow \lfloor \frac{l+r}{2} \rfloor$ 
    if  $K = A[m]$  then
      return  $m$ 
    else if  $K < A[m]$  then
       $r \leftarrow m - 1$ 
    else
       $l \leftarrow m + 1$ 
  return  $-1$ 

```

To find an item, we look for it in the middle. If we do not find it, we look only to the left (if the target is smaller than the middle element) or to the right (if the target is larger than the middle element).

We will not look in detail at the recurrence or its solution right now, just note that this is a very fast algorithm: requiring  $\Theta(\log n)$  comparisons in the worst case.

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## Fake Coin Problem

The text discusses a version of the *fake coin problem* where we are given a set of  $n$  coins, one of which is a fake. We know it is a fake because it is lighter than the legitimate coins. To solve the problem we are able to use a balance scale, and on each side of the scale we can put 1 or more coins to compare the weights.

Discussion:

- a naive approach: weigh pairs of coins until we find the one that's lighter than the others
- a reduce-by-constant-factor approach where we weigh equal numbers of coins (half at a time), narrowing down which half contains the fake until we find it
- a reduce-by-constant-factor approach where we weigh 3 piles of coins at a time, narrowing down which third contains the fake until we find it
- does it make sense to divide into more than 3 piles?

The text has a few additional examples that we may encounter later.