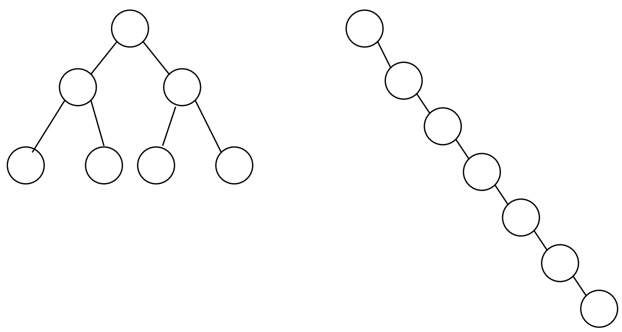
Computer Science 385
Design and Analysis of Algorithms
Siena College
Spring 2024

Topic Notes: AVL Trees

Balanced Trees

The efficiency of the operations on binary search trees (and other binary trees, for that matter) relies on them being reasonably *balanced trees*. That is, the height is proportional to $\log n$, not n.



Just having the same height on each child of the root is not enough to maintain a $\Theta(\log n)$ height for a binary tree.

We can define a *balance condition*, some set of rules about how the subtrees of a node can differ.

Maintaining a perfectly strict balance (minimum height for the given number of nodes) is often too expensive. Maintaining too loose a balance can destroy the $\Theta(\log n)$ behaviors that often motivate the use of tree structures in the first place.

For a strict balance, we could require that all levels except the lowest are full.

How could we achieve this? Let's think about it by inserting the values 1,2,3,4,5,6,7 into a BST and seeing how we could maintain strict balance.

First, insert 1:

```
1
```

Next, insert 2:

```
1
 \
  2
```

We're OK there. But when we insert 3:

```
   1
    \
     2
      \
       3
```

we have violated our strict balance condition. Only one tree with these three values satisfies the condition:

```
   2
  / \
 1   3
```

We will see how to "rotate" the tree to achieve this shortly.

Now, add 4:

```
   2
  / \
 1   3
      \
       4
```

Then add 5:

```
   2
  / \
 1   3
      \
       4
        \
         5
```

Again, we need to fix the balance condition. Here, we can apply one of these rotations on the right subtree of the root:

```
   2
  / \
 1   4
    / \
   3   5
```

Now, add 6:

```
    2
   / \
  1   4
     / \
    3   5
         \
          6
```

It's not completely obvious how to fix this one up, and we won't worry about it just now. We do know that after we insert the 7, there's only one permissible tree:

```
       4
      / \
    2     6
   / \   / \
  1   3 5   7
```

So maintaining strict balance can be very expensive. The tree adjustments can be more expensive than the benefits.

There are several options to deal with potentially unbalanced trees without requiring a perfect balance.

1. *Red-black trees* – nodes are colored red or black, and place restrictons on when red nodes and black nodes can cluster.
2. *AVL Trees* - Adelson-Velskii and Landis developed these in 1962. We will look at these.
3. *Splay trees* – every reference to a node causes that node to be relocated to the root of the tree.

   This is very unusual! We have a `contains()` operation that actually modifies the structure.

   This works very well in cases where the same value or a small group of values are likely to be accessed repeatedly.
4. *2-3 Trees* – tree nodes can hold more than one key – described in the Levitin Algorithms text.

AVL Trees

We consider *AVL Trees*, developed by and named for Adelson-Velskii and Landis, who invented them in 1962.

The balance condition for AVL trees (the *AVL condition)*: the heights of the left and right subtrees of any node can differ by at most 1.

To see that this is less strict than perfect balance, let's consider two trees:

```
     5
   /   \
  2     8
 / \   /
1   4 7
   /
  3
```

This one satisfies the AVL condition (to decide this, we check the heights at each node), but is not perfectly balanced since we could store these 7 values in a tree of height 2.

But...

```
     7
   /   \
  2     8
 / \
1   4
   / \
  3   5
```

This one does not satisfy the AVL condition – the root node violates it!

So the goal is to maintain the AVL balance condition each time there is an insertion (we will ignore deletions, but similar techniques apply).

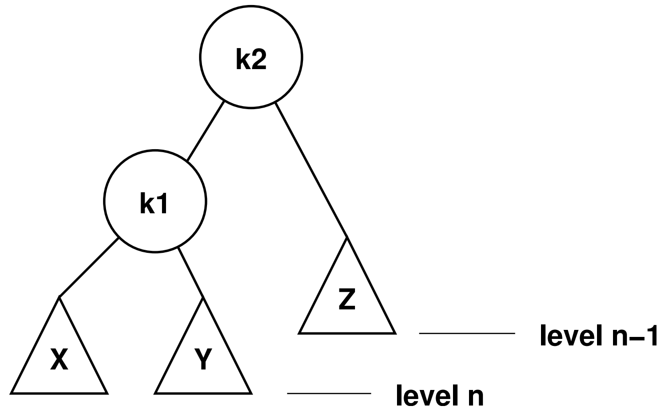
When inserting into the tree, a node in the tree can become a violator of the AVL condition. Four cases can arise which characterize how the condition came to be violated. Let's call the violating node A.

1. Insertion into the left subtree of the left child of A.
2. Insertion into the right subtree of the left child of A.
3. Insertion into the left subtree of the right child of A.
4. Insertion into the right subtree of the right child of A.

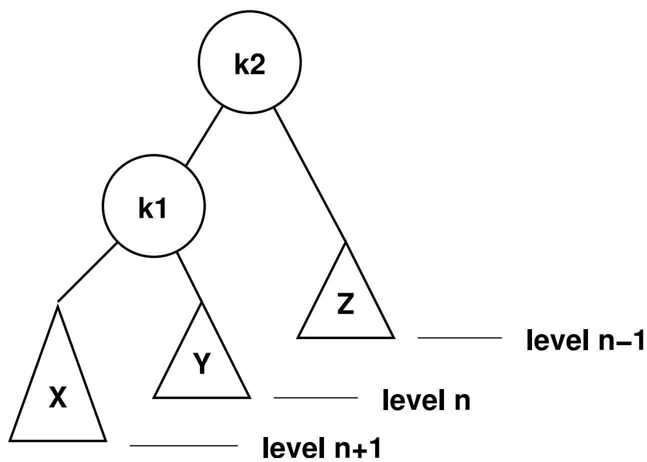
In reality, however, there are only two really different cases, since cases 1 and 4 and cases 2 and 3 are mirror images of each other and similar techniques apply.

First, we consider a violation of case 1.

We start with a tree that satisfies AVL:



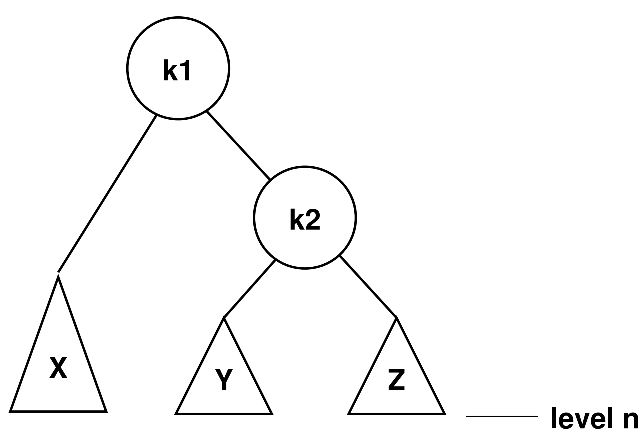
After an insert, the subtree X increases in height by 1:



So now node k_2 violates the balance condition.

We want to perform a *single rotation* to obtain an equivalent tree that satisfies AVL.

Essentially, we want to switch the roles of k_1 and k_2, resulting in this tree:



For this insertion type (left subtree of a left child – case 1), this rotation has restored balance.

We can think of this like you have a handle for the subtree at the root and gravity determines the tree.

If we switch the handle from k_2 to k_1 and let things fall where they want (in fact, must), we have rebalanced.

Consider insertion of 3,2,1,4,5,6,7 into an originally empty tree.

Insert 3:

```
    3
```

Insert 2:

```
    3
   /
  2
```

Insert 1:

```
    3                  2
   /                  / \
  2          --->    1   3
 /
1
```

Here, we had to do a rotation. We essentially replaced the root of the violating subtree with the root of the taller of its children.

Now, we insert 4:

```
  2
 / \
1   3
     \
      4
```

Then insert 5:

```
  2
 / \
1   3  <-- AVL violated here (case 4)
```

```
         \
          4
           \
            5
```

and we have to rotate at 3:

```
      2
     / \
    1   4
       / \
      3   5
```

Now insert 6:

```
      2   <-- AVL violated here (case 4)
     / \
    1   4
       / \
      3   5
           \
            6
```

Here, our rotation moves 4 to the root and everything else falls into place:

```
      4
     / \
    2   5
   / \   \
  1   3   6
```

Finally, we insert 7:

```
      4
     / \
    2   5  <-- AVL violated here (case 4 again)
   / \   \
  1   3   6
           \
            7
```

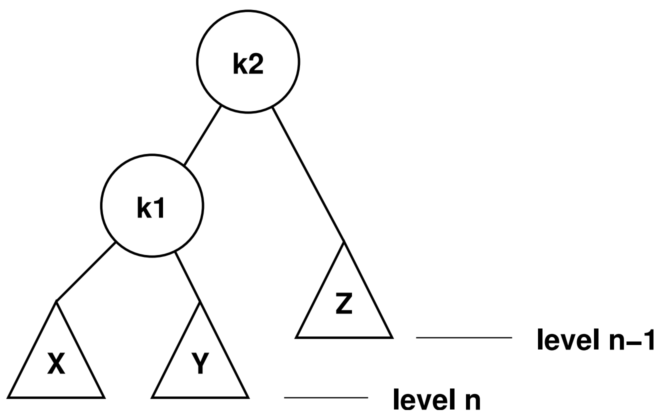
```
      4
    /   \
   2     6
  / \   / \
 1   3 5   7
```

We achieve perfect balance in this case, but this is not guaranteed in general.

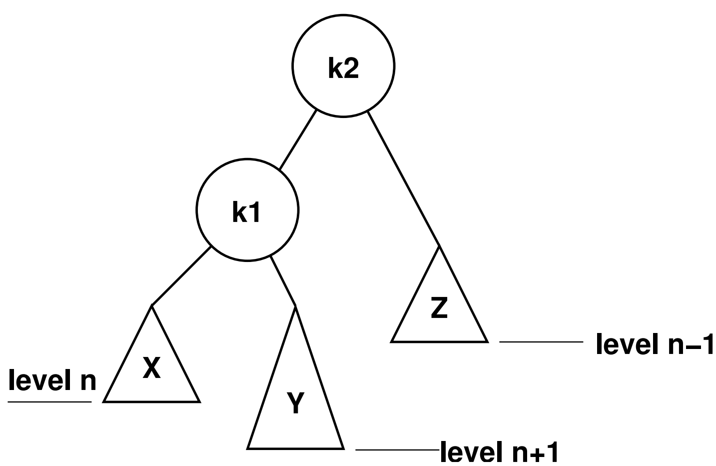
This example demonstrates the application of cases 1 and 4, but not cases 2 and 3.

Here's case 2:

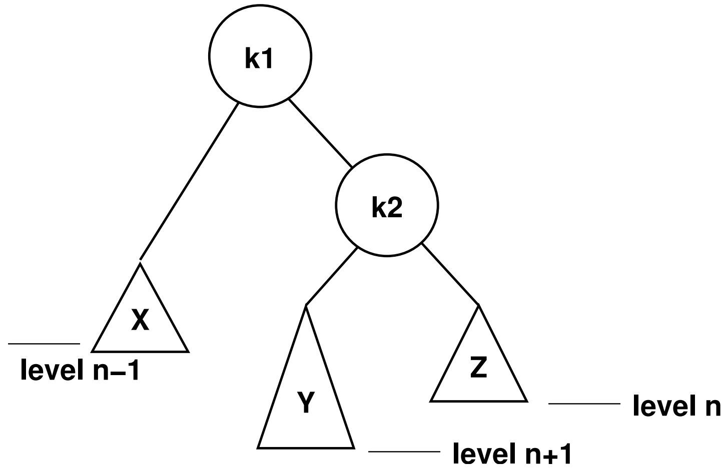
We start again with the good tree:



But now, our inserted item ends up in subtree Y:

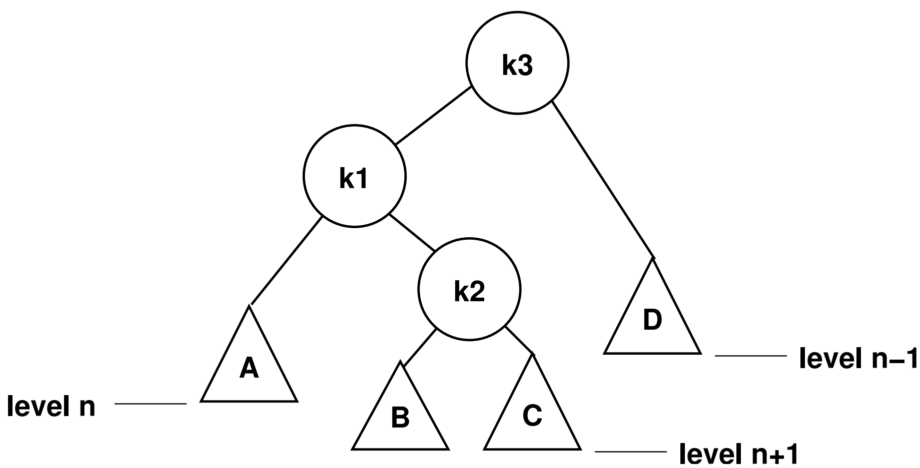


We can attempt a single rotation:



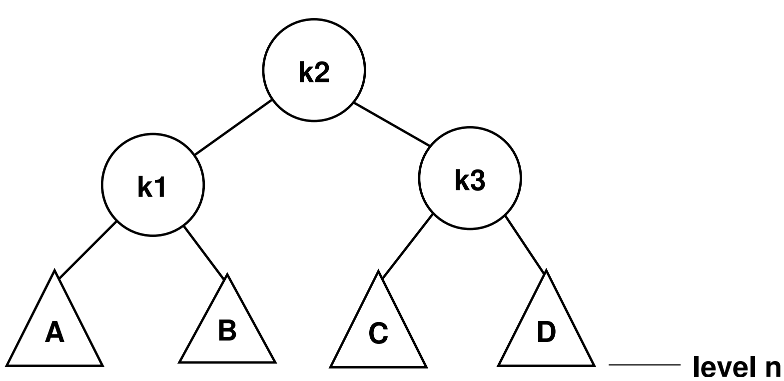
This didn't get us anywhere. We need to be able to break up Y.

We know subtree Y is not empty, so let's draw our tree as follows:



Here, only one of B or C is at level $n + 1$, since it was a single insert below k_2 that resulted in the AVL condition being violated at k_3 with respect to its shorter child D.

We are guaranteed to correct it by moving D down a level and both B and C up a level:



We're essentially rearranging k_1, k_2, and k_3 to have k_2 at the root, and dropping in the subtrees in the only locations where they can fit.

In reality, only one of B and C is at level n – the other only descends to level $n - 1$.

Case 3 is the mirror image of this.

To see examples of this, let's pick up the previous example, which had constructed a perfectly-balanced tree of the values 1–7.

```
      4
    /   \
   2     6
  / \   / \
 1   3 5   7
```

At this point, we insert a 16, then a 15 to get:

```
      4
    /   \
   2     6
  / \   / \
 1   3 5   7
            \
             16
            /
          15
```

Node 7 violates AVL and this happened because of an insert into the left subtree of its right child. Case 3.

So we let k_1 be 7, k_2 be 15, and k_3 be 16 and rearrange them to have k_2 at the root of the subtree, with children k_1 and k_3. Here, the subtrees A, B, C, and D are all empty.

We get:

```
      4
    /   \
   2     6
  / \   / \
 1   3 5   15
          /  \
         7    16
```

Now insert 14.

```
        4
      /   \
    2      6
   / \    / \
  1   3  5   15
            /  \
           7    16
            \
             14
```

This violates AVL at node 6 (one child of height 0, one of height 2).

This is again an instance of case 3: insertion into the left subtree of the right child of the violating node.

So we let k_1 be 6, k_2 be 7, and k_3 be 15 and rearrange them again. This time, subtrees A is the 5, B is empty, C is the 14, and D is the 16.

The *double rotation* requires that 7 become the root of that subtree, the 6 and the 15 its children, and the other subtrees fall into place:

```
         4
       /   \
    2        7
   / \      / \
  1   3   6    15
         /    /  \
        5   14    16
```

Insert 13:

```
         4
       /   \
    2        7
   / \      / \
  1   3   6    15
         /    /  \
        5   14    16
           /
         13
```

What do we have here? Looking up from the insert location, the first element that violates the balance condition is the root, which has a difference of two between its left and right child heights.

Since this is an insert into the right subtree of the right child, we're dealing with case 4. This requires just a single rotation, but one done all the way at the root. We get:

```
             7
           /   \
       4           15
     /   \       /   \
   2       6   14     16
  / \     /   /
 1   3   5  13
```

Now adding 12:

```
             7
           /   \
       4           15
     /   \       /   \
   2       6   14     16
  / \     /   /
 1   3   5  13
           /
          12
```

The violation this time is at 14, which is a simple single rotation (case 1):

```
             7
           /   \
       4           15
     /   \       /   \
   2       6   13     16
  / \     /   /   \
 1   3   5  12   14
```

Inserting 11:

```
             7
           /   \
       4           15
     /   \       /   \
   2       6   13     16
  / \     /   /   \
 1   3   5  12   14
           /
          11
```

Here, we have a violation at 15, case 1, so another single rotation there, promoting 13:

```
            7
          /   \
       4         13
     /   \      /  \
   2      6   12    15
  / \     /  /     /  \
 1   3   5  11   14     16
```

(Almost done)

Insert 10:

```
            7
          /   \
       4         13
     /   \      /  \
   2      6   12    15
  / \     /  /     /  \
 1   3   5  11   14     16
           /
          10
```

The violator here is 12, case 1:

```
            7
          /   \
       4         13
     /   \      /   \
   2      6   11     15
  / \     /  / \    /  \
 1   3   5  10  12 14  16
```

Then we finally add 8 (no rotations needed) then 9:

```
            7
          /   \
       4         13
     /   \      /   \
   2      6   11     15
  / \     /  / \    /  \
 1   3   5  10  12 14  16
           /
          8
           \
            9
```

Finally we see case 2 and do a double rotation with 8, 9, and 10 to get our final tree:

```
           7
         /   \
     4          13
   /   \      /    \
  2      6   11      15
 / \    /   / \    /   \
1   3  5   9  12 14   16
          / \
         8  10
```

This tree is not strictly balanced – we have a hole under 6's right child, but it does satisfy AVL.

You can think about how we might implement an AVL tree, but we will not consider an actual implementation. However, AVL insert operations make excellent exam questions, so keep that in mind when preparing for the final.

The whole point of considering AVL trees is to maintain a reasonable balance, and hopefully, a tree height that looks like $\log n$. We will not do a detailed analysis, but the height n of an AVL tree is guaranteed to satisfy the inequality:

$$\lfloor \log_2(n+1) \rfloor \leq h < 1.44 \log_2(n+2) - 0.328.$$

We have log factors on both sides, leading to $\Theta(\log n)$ worst case behavior of search and insert operations.